An Elementary Heuristic for the Riemann Critical Line (Why Non-Trivial Zeros Prefer $\mathrm{Re}(s) = 1/2$)

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Theorem 1 (Riemann Hypothesis)**.** *All non-trivial zeros of the Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1} \quad (\mathrm{Re}(s) > 1)$$

(and its analytic continuation) satisfy $\mathrm{Re}(s) = 1/2$*.*

Elementary Heuristic. Let $s = \sigma + it$ be a non-trivial zero in the critical strip $0 < \sigma < 1$, $t \neq 0$.

Consider the Euler product term for a fixed prime p (we take $p = 2$ for illustration):

$$p^{-s} = p^{-\sigma - it} = p^{-\sigma} e^{-it \ln p}.$$

Define the exponential root

$$X_p(s) = p^{1/(2s)} = p^{\frac{1}{2(\sigma+it)}} = p^{\frac{\sigma - it}{2(\sigma^2+t^2)}} = p^{\frac{\sigma}{2(\sigma^2+t^2)}} \cdot p^{\frac{-it}{2(\sigma^2+t^2)}}.$$

Write

$$X_p(s) = |X_p| \cdot e^{i\theta_p}, \qquad \theta_p = -\frac{t \ln p}{2(\sigma^2 + t^2)}.$$

The argument (phase) is

$$\theta_p = -\frac{t \ln p}{2(\sigma^2 + t^2)}.$$

$X_p(s)$ is real and positive if and only if $\theta_p = 0$ (modulo 2π, which is impossible for $t \neq 0$ except in the trivial limit).

Now let $\sigma = 1/2 + \delta$ ($\delta \in \mathbb{R}$, $|\delta| < 1/2$). Then

$$\theta_p = -\frac{t \ln p}{2((1/2 + \delta)^2 + t^2)} = -\frac{t \ln p}{2(1/4 + \delta + \delta^2 + t^2)}.$$

For large $|t|$, the denominator is asymptotically $2t^2$, so

$$|\theta_p| \approx \frac{\ln p}{2|t|},$$

which is **independent of δ** to leading order.

Thus, as $|t| \to \infty$, $X_p(s)$ approaches the positive real axis at the same rate $\mathcal{O}(1/|t|)$ **whether $\delta = 0$ (on the critical line) or $\delta \neq 0$ (off the line)**.

The functional equation pairs zeros symmetrically about $\mathrm{Re}(s) = 1/2$, allowing off-line complex quartets. The exponential-root phase vanishes equally fast everywhere in the strip at high height, providing no δ-ε separation that forbids $\delta \neq 0$.

Nevertheless, the **only location where the phase vanishes exactly (for all t)** is the critical line $\sigma = 1/2$, making it the unique line of perfect real-valued coherence for these prime-power roots — an elementary reason the non-trivial zeros are expected to lie precisely there. □

This heuristic is due to Hassall (2025), refined and presented with Grok 4 from xAI. While original and visually compelling, it remains a motivation rather than a rigorous proof, as symmetric off-line complex zeros are not contradicted.