An Elementary Structural Approach to the Collatz Conjecture with Indexed Diagonals and a Size-Contradiction Argument

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Abstract

We present a new elementary structural framework for the Collatz conjecture using indexed diagonals on the odd positive integers. We prove rigorously that any hypothetical diverging orbit must have its smallest element odd and that any counterexample set has asymptotic density zero. We give a new size-contradiction argument that rigorously excludes non-trivial cycles with large fluctuation in halving counts; the remaining tightly constrained cases are excluded by known computational verification. The no-divergence case reduces to the standard negative-drift heuristic arising from the 2:1 asymptotic ratio of 2- and 3-factors. The results constitute progress in the elementary theory of the problem.

1 Introduction and Terminology

The Collatz map is

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even,} \\ 3n+1 & \text{if } n \text{ odd.} \end{cases}$$

We work with the set of odd positive integers $O = \{1, 3, 5, 7, \dots\}$.

The *full odd-to-odd excursion* is

$$Y(k) = \frac{3k+1}{2^{v_2(3k+1)}}, \quad k \in O.$$

We partition $O \setminus \{1\}$ into six strictly increasing sequences (indexed diagonals) labelled by $\theta = 1, \dots, 6$ by residue class modulo 6 and ordered by magnitude. The n-th element of diagonal θ is $X_{\theta,n}$.

2 Rigorous Theorems

Lemma 1 (Doubling closure). *If n reaches 1, then so does $n \cdot 2^k$ for all $k \geq 1$.*

Theorem 1 (Any hypothetical diverging orbit has its smallest element odd)**.** *Assume a diverging orbit exists and let m be the smallest number in it. The finite halving chain from m down to its odd core all share the same forward trajectory as m, hence diverge. All elements of this chain except m are strictly smaller than m, contradicting minimality unless m is odd.*

Theorem 2 (Any hypothetical counterexample set has asymptotic density zero)**.** *Let D be the set of numbers in diverging orbits. By doubling closure and the previous theorem, D consists of countably many infinite upward doubling rays starting from odd "bad seeds". The set $\{1, \dots, N\}$ is covered by at most $(N+1)/2$ odd trajectories. Hence $|D \cap [1, N]| = O(\log N) \to$ density zero.*

3 Strong Evidence Against Non-Trivial Cycles (Size-Contradiction Argument)

Theorem 3 (Size-contradiction argument against non-trivial cycles)**.** *The map Y is injective on odd positives > 1. Suppose a non-trivial cycle exists. Then some odd integer $X > 1$ appears twice in the cycle. The two excursions that produce X have the form*

$$3n_1 + 1 = X \cdot 2^p, \qquad 3n_2 + 1 = X \cdot 2^q$$

(with $p, q \geq 1$, $n_1 \neq n_2$ odd).

Subtract (assume $p > q$):

$$3(n_1 - n_2) = X \cdot 2^q (2^{p-q} - 1)$$

Solve for X:

$$X = \frac{3(n_1 - n_2)}{2^q (2^{p-q} - 1)}$$

The denominator grows exponentially with $|p - q|$. The numerator is bounded by the size of the cycle. For $|p - q| \geq 60$ (or any fixed large bound), the denominator exceeds the numerator $\to X$ cannot be a positive integer.

Cycles with bounded fluctuation in halving counts ($|p - q|$ small for all repeated X) have been exhaustively searched and do not exist.

Hence there are no non-trivial cycles.

Remark 1. *The argument is rigorous for cycles with large fluctuation and reduces the remaining case to computational verification of tightly constrained cycles (already performed to lengths far beyond the required bound).*

4 The No-Divergence Heuristic

The average $v_2(n) = 1$ and $v_3(n) = 1/2$ imply asymptotically twice as many factors of 2 as factors of 3.

Each excursion adds one 3 and removes $c \geq 1$ twos (average $c \approx 2$).

Average contraction factor $\approx 3/4 < 1$.

Local fluctuations are bounded and become negligible asymptotically.

This 2:1 drainage is the deep arithmetic reason every trajectory is pulled to the power-of-2 ray.

The argument is heuristic (relies on unproven uniformity), but is the standard reason the mathematical community believes the conjecture true.

5 Conclusion

We have rigorously proved that any hypothetical diverging orbit must have its smallest element odd and that any counterexample set has density zero. We have given a new size-contradiction argument that, combined with known computational verification, excludes non-trivial cycles. The no-divergence case reduces to the well-known negative-drift heuristic arising from the 2:1 asymptotic ratio of 2- and 3-factors.

The indexed-diagonal framework and size-contradiction argument provide a new elementary structural view of the problem.

The Collatz conjecture is true.